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Third Semester M.Sc. Degree Examination, Dec. 2015
(CBCS)
MATHEMATICS
M 301 T : Differential Geometry

Time : 3 Hours

Max. Marks : 70

Instructions : *1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Define :

i) Natural co-ordinate function

ii) Tangent vector

iii) Natural frame field.

With usual notations on E^3, prove that $V = \sum_{i=1}^{3} v_i U_i$.

b) Let $V_1 = U_1 - x U_3$, $V_2 = U_2$ and $V_3 = x U_1 + U_3$. Prove that the vectors $V_1(P)$, $V_2(P)$, $V_3(P)$ are linearly independent at each point of E^3.

c) Define :

i) Curves in E^3.

ii) Reparametrization on E^3.

Let α be a curve in E^3 and f be a differentiable function on E^3. Then prove that $\alpha'(t)[f] = \frac{d}{dt}(f \circ \alpha)(t)$. **(5+4+5)**

2. a) If ϕ is a 1-form on E^3, then show that $\phi = \sum f_i \, dx_i$, where $f_i = \phi(u_i) = \phi . u_i$.

b) Let f and g be functions, ϕ and ψ are 1-forms. Then show that

$d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$.

c) For any function f, show that $d(df) = 0$. Deduce that $d(f\,dg) = df \wedge dg$. **(4+6+4)**

P.T.O.

3. a) Define a derivative map. Let $F = (f_1, f_2, \ldots, f_m)$ be a mapping from E^n to E^m. If v_p is a tangent vector to E^n at P, then show that $F*(v_p) = (v_p[f_1], v_p[f_2], \ldots, v_p[f_m])$ at F(P).

b) With usual notations, prove that $\begin{bmatrix} T^1 \\ N^1 \\ B^1 \end{bmatrix} = \begin{bmatrix} 0 & K & 0 \\ -K & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$.

c) Let β be a unit speed curve in E^3 with $k > 0$. Then show that β is a plane curve if and only if $\tau = 0$. **(5+5+4)**

4. a) Define a covariant derivative. If W is a vector field with constant length $\|W\|$, prove that for any vector field V, the covariant derivative $\nabla_v W$ is every where orthogonal to W.

b) Define connection forms. Let E_1, E_2, E_3 be a frame field on E^3. For each tangent vector v to E^3 at P, let $w_{ij}(v) = \nabla_v E_i \cdot E_j(P), (1 \leq i, j \leq 3)$. Then show that each w_{ij} is a 1-form and $w_{ij} = -w_{ji}$.

c) Define an isometry on E^3. If F and G are isometries of E^3, then show that the composite mapping $G \circ F$ is also an isometry of E^3. **(5+5+4)**

5. a) Define a surface in E^3. Use the definition to show that a sphere of unit radius with centre origin is a surface.

b) Show that the function $\mathcal{X} : D \to E^3$ defined by $\mathcal{X}(u, v) = (u^2, uv, v^2)$, $u > 0, v > 0$ is a proper patch.

c) Show that the ellipsoid $M : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is a simple surface. **(4+5+5)**

6. a) Show that a mapping $\mathcal{X} : D \to E^3$ is regular if and only if the (u, v) partial derivatives $\mathcal{X}_u(d)$ and $\mathcal{X}_v(d)$ are linearly independent for all $d \in D C E^2$.

b) Obtain parametrization of surface of revolution.

c) Let $F: M \to N$ be a mapping of surfaces and F^* be its pull back function. Prove the following for p-forms (p = 0, 1, 2) ξ and η

i) $F^*(\xi + \eta) = F^*\xi + F^*\eta$

ii) $F^*(\xi \wedge \eta) = F^*\xi \wedge F^*\eta$ **(5+5+4)**

7. a) Define shape operator of a surface in E^3. Find the shape operator of the following surfaces :

i) Sphere of radius r

ii) A cylinder.

b) Define normal curvature of a surface. If 'P' is an umbilic point of a surface M in E^3, then prove that the shape operator S at P is just a scalar multiplication by $K = K_1 K_2$. **(8+6)**

8. a) If $\mathcal{X}$ is a patch in a surface M in E^3, then prove that the Gaussian and mean curvatures of M are given by $K = \frac{ln - m^2}{EG - F^2}$, $H = \frac{Gl + En - 2Fm}{2(EG - F^2)}$ where $l = U.\mathcal{X}_{uu}$, $M = U.\mathcal{X}_{uv}$, $n = U.\mathcal{X}_{vv}$, U is a unit vector field on M and $E = \mathcal{X}_u.\mathcal{X}_u$, $F = \mathcal{X}_u.\mathcal{X}_v$, $G = \mathcal{X}_v.\mathcal{X}_v$.

b) Find the Gaussian curvature of the ellipsoid $g : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

c) Find the geodesics in a cylinder. **(6+4+4)**

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III Semester M.Sc. Degree Examination, December 2016
(CBCS)
MATHEMATICS
M 301T : Differential Geometry

Time : 3 Hours Max. Marks : 70

Instructions : *1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Define directional derivative of a differentiable function by a tangent vector. If $V_P = (v_1, v_2, v_3)_p$ is a tangent vector to E^3 at p and f is a real valued differentiable function on E^3 then show that the directional derivative $V_p[f] = \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i}(p)$. Use it to prove $U_i[f] = \frac{\partial f}{\partial x_i}$ for a natural frame field (U_1, U_2, U_3) on E^3. 7

   b) If $V = x\,U_1 - y^2U_3$, $f = x^2y + z^3$, and $g = xy$ then compute V[f], V[g] and V[fg]. 7

2. a) Define reparametrisation of a curve. Show that every regular curve has a unit speed reparametrisation. 6

   b) Reparametrise the curve $\alpha(t) = (a\cos t, a\sin t, bt)$ by its arc length, where $b > 0$. 5

   c) Evaluate the 1-form $\phi = x^2 dx - y^2 dz$ on the vector field $V = xU_1 + yU_2 + zU_3$. 3

3. a) If ϕ and ψ are any two 1-forms on E^3 then prove that $d(\phi \wedge \psi) = (d\phi \wedge \psi - \phi \wedge d\psi)$. 6

   b) If β is a unit speed curve with constant curvature $k > 0$ and torsion zero then prove that β is part of a circle of radius $1/k$. 4

   c) Show that the curve $\beta(s) = \left(\frac{4}{5}\cos s, 1 - \sin s, \frac{3}{5}\cos s\right)$ is a circle. 4

4. a) Let $V = (1, -1, 2)$, $P = (1, 3, -1)$ and $W = x^2U_1 + yU_2$. Then compute $\nabla_{V_p} W$. 4

   b) Obtain the connection forms for a cylindrical frame field. 5

   c) Prove that a translation, a rotation and an orthogonal transformation are isometries. 5

P.T.O.

5. a) Define a proper patch. If f is a real valued differentiable function on E^3 then prove that the map $X : D \subset E^2 \to E^3 : X(u, v) = (u, v, f(u, v))\ \forall (u, v) \in D$ is a proper patch. 6

b) Is cylinder in E^3, a surface ? Justify. 4

c) If $M : g(x, y, z) = C$ is a surface in E^3 then show that the gradient vector field $\nabla g = \sum \frac{\partial g}{\partial x_i} U_i$ is a non vanishing normal vector field on E^3. 4

6. a) Obtain parametrisation of the following :

i) A cylinder in E^3.

ii) Entire surface obtained by revolving the curve $C : y = \cosh x$ around x – axis. 6

b) With usual notations prove

i) $d(F^*\xi) = F^*(d\xi)$, for any 1 – form ξ

ii) $X^*(\phi) = \phi(X_u)du + \phi(X_v)dv$ for any 1 – form ϕ.

iii) $X^*(d\phi) = \left(\frac{\partial}{\partial u}(\phi(X_v)) - \frac{\partial}{\partial v}(\phi(X_u))\right)du\,dv$. 8

7. a) Define shape operator of a surface at a point. Show that it is a linear operator. 5

b) With usual notations prove $K = k_1 k_2$, $H = \frac{1}{2}(k_1 + k_2)$. 4

c) If V and W are linearly independent tangent vectors at a point P of $M \subset E^3$ then prove that

i) $S(V) \times S(W) = K(P)\, V \times W$.

ii) $SV \times W + V \times SW = 2\, H(P)\, V \times W$. 5

8. a) Compute the Gaussian, the mean curvatures and hence the principal curvatures k_1, k_2 for the surface $X(u, v) = (u \cos v, u \sin v, bv)$, $b \neq 0$. 4

b) Let α be a regular curve in a surface M in E^3 and let U be a unit normal vector field restricted to α. Then prove that the curve α is principal if and only if U' and α' are collinear at each point. 6

c) Determine the geodesics in

i) a plane ii) a sphere 4

III Semester M.Sc. Degree Examination, December 2014
(RNS) (Y2K11 Scheme)
MATHEMATICS
M303 : Differential Geometry

Time : 3 Hours Max. Marks : 80

Instructions : 1) *Answer **any five** questions choosing atleast **two** from **each** Part.*
2) ***All** questions carry **equal** marks.*

PART – A

1. a) Define directional derivative of a differentiable function on E^3.
If $V_p = (v_1, v_2, v_3)_p$ in any tangent vector to E^3 at $P \in E^3$, then prove that for any real valued differentiable function f on E^3, directional derivative of f by V_p in $V_p[f] = \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_2}(p)$.
Use the above the formula to compute $Vp[f]$ for $f = e^x \cos y$, $Vp : P = (2, 0, -1), V = (2, -1, 3)$. 10

b) Let $h(s) = \log s$ on $J : s > 0$. Reparametrize the curve $\alpha(t) = (e^t, e^{-t}, \sqrt{2}\, t$ using h. Verify the formula $B'(s) = \alpha'(h(s)) \frac{dh}{ds}(s)$, were B in a reparametrisation of α by h. 6

2. a) Let $f(x, y, z) = (x^2 - 1)\, dy + (y^2 + 2)\, z$. Find the 1 – form df and evaluate it an $V_p : V = (1, 2, -3), P = (0, -2, 1)$. 5

b) Let $\phi = yzdx + dz$, $\psi = \sin z\, dz + \cos z\, dy$ be two 1 – forms on E^3. Then compute $\phi \wedge \psi$ and $d(\phi \wedge \psi)$. Verify the formula $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$. 6

c) Let $F : E^3 \to E^3$ be a mapping defined by $F(x, y, z) = (x \cos y, x \sin y, z)$. Then compute $F_{*p} V_p$ for $Vp : V = (2, -1, 3), P = (0, 0, 1)$. 5

P.T.O.

3. a) Derive Frenet formulae for a unit speed curve. 5

b) Compute the Frenet apparatus T, N, B, K, T for a unit speed curve

$\beta(s) = \left(\frac{4}{5}\cos s,\ 1 - \sin s, \frac{-3}{5}\cos s\right)$ and show that it is a circle. 5

c) With usual notations derive the formula $\nabla v_p W = \sum V_p[w_i]\ U_i\ (p)$, where $W = (w_1, w_2, w_3)$ is a vector field on E^3. Use it to compute $\nabla v_p W$ for $W = x^2 V_1 + y V_2$ and $V_p : V = (1, -1, 2)$ and $P = (1, 3, -1)$. 6

4. a) Let E_1, E_2, E_3 be a frame field on E^3 with the attitude matrix $A = (a_{ij})$. Then show that the matrix of connection forms $W = (W_{ij})$ of E_1, E_2, E_3 is given by $W = (dA)\ A^t$, where dA is the differential and A^t is the transpose of A. Use the formula to compute connection forms of a cylindrical frame field. 9

b) Prove the following :

i) an orthogonal transformation on E^3 is an isometry.

ii) an isometry $F : E^3 \to E^3$ with $F(0) = 0$ is an orthogonal transformation. 7

PART – B

5. a) Define :

i) Proper patch

ii) Simple surface.

If $X : E^2 \to E^3$ is defined by X (u, v) = (u + v, u – v, uv), then prove that X is a proper patch in E^3 and its image is the surface $z = \frac{x^2 - y^2}{4}$. 6

b) Let g be a real valued differentiable function on E^3 and 'c' be a real number. Show that the subset $M = \{(x, y, z) \in E^3 / g(x, y, z) = c\}$ is a surface in E^3 if $dg \neq 0$ at any point of M. Hence deduce that a sphere is a surface in E^3. (6+4)

6. a) Obtain parametrization of a cylinder. 4

b) Let P be any point in a surface M and X be a patch in M with $P = X(u_0, v_0)$. Show that a tangent vector V_p to E^3 at P is tangent to M at P if and only if V_p is a linear combination of $X_u (u_0, v_0)$ and $X_u (u_0, v_0)$. 6

c) With usual notations, prove

i) $F^*(\xi \wedge \eta) = F^*(\xi) \wedge F^*\eta$

ii) $F^*(d\xi) = d(F^*\xi)$. 6

7. a) Define shape operator of a surface at a point. Find the shape operator of the saddle surface at (0, 0, 0). 6

b) Show that every point on a sphere is a umbilic point. 6

c) With usual notations prove $K = k_1 k_2$, $H = y_2 (k_1 + k_2)$. 4

8. a) Prove that the Gaussian and mean curvatures of a surface are given by 4

$$K = \frac{\begin{vmatrix} SV.V & SV.W \\ SW.V & SW.W \end{vmatrix}}{\begin{vmatrix} V.V & V.W \\ W.V & W.W \end{vmatrix}}, \qquad H = \frac{\begin{vmatrix} SV.V & SV.W \\ W.V & W.W \end{vmatrix} + \begin{vmatrix} V.V & V.W \\ SW.V & SW.W \end{vmatrix}}{2\begin{vmatrix} V.V & V.W \\ W.V & W.W \end{vmatrix}}$$

b) If X is a patch in a surface M in E^3, then prove that the fundamental magnitudes l, m, n are given by $l = U.X_{uu}$, $M = U.X_{uv}$, $n = U.X_{vv}$, where U is a unit normal vector field on M. Hence compute the Gaussian and mean curvatures of $X(u, v) = (u \cos v, u \sin v, bv)$, $b \neq 0$. 7

c) Determine the geodesics in 5

i) plane

ii) sphere.

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III Semester M.Sc. Degree Examination, January 2013
(R.N.S.) (Y2K11 Scheme)
MATHEMATICS
M 303 : Differential Geometry

Time : 3 Hours Max. Marks : 80

Instructions : *1. Answer **any five** questions choosing atleast **two** from **each** Part.*
*2. **All** questions carry **equal** marks.*

PART – A

1. a) If $V_1 = U_1 - xU_3$, $V_2 = U_2$ and $V_3 = xU_1 + U_3$ then prove that :
   i) $V_1(P)$, $V_2(P)$ and $V_3(P)$ are linearly independent at each $P \in E^3$.
   ii) The vector field $xU_1 + yU_2 + zU_3$ can be expressible as a linear combination of V_1, V_2, V_3. **8**

   b) Compute $V_p[f]$ for $V_p : V = (2, 1, -3)$, $P = (2, 0, -1)$ and $f = y^2 z$. **4**

   c) Prove the identity $V = \Sigma\, V[x_i]\, U_i$ for any vector field V, where x_1, x_2, x_3 are natural coordinate functions. **4**

2. a) Let α be a curve in E^3 and let f be a differentiable function on E^3. Then prove that $\alpha'(t)[f] = \frac{d}{dt}(f \circ \alpha)(t)$. **6**

   b) Let $h(s) = \log s$ on J and $S > 0$. Reparametrize $\alpha(t) = (e^t, e^{-t}, \sqrt{2}\, t)$ using h. **4**

   c) Evaluate the 1-form $(z^2 - 1)\, dx - dy + x^2\, dz$ on the tangent vector $V_p : V = (1, 2, -3)$, $P = (0, -2, 1)$. **6**

3. a) If $F : E^n \to E^m$ is a mapping with $F = (f_1, f_2, \ldots, f_m)$, then prove that for any tangent vector V_p, $F_{*p} V_p = (V_p[f_1], V_p[f_2], \ldots, V_p[f_m])_{F(p)}$. Hence deduce that $F_{*p}(U_j(P)) = \sum_{i=1}^{M} \frac{\partial f_i}{\partial x_j}(P)\, U_i(F(P))$, $J = 1, 2, \ldots, n$. **8**

   b) Compute the Frenet apparatus T, N, B, K, τ for a curve $\beta(S) = \left(a \cos \frac{s}{c}, b \sin \frac{s}{c}, b \frac{s}{c}\right)$, where $C = \sqrt{a^2 + b^2}$. **8**

P.T.O.

4. a) If α is a regular curve with speed function v, then prove that the velocity and acceleration of α are given by $\alpha' = vT, \alpha'' = \frac{dv}{dt}T + kv^2N$. 4

b) Let $V = (y - x)U_1 + xyU_3$,
$W = x^2 U_1 + yzU_3$. Compute $\nabla_v W$. 5

c) Prove that for an isometry F of E^3 there exist a unique translation T and a unique orthogonal transformation C such that F = TC. 7

PART – B

5. a) Define a patch in E^3. Show that for a real valued differentiable function on a non empty open set D of E^2, the function $X : D \to E^3$ defines by $X(u, v) = (u, v, f(u, v))$ is a proper patch in E^3. 7

b) Verify whether the function $X(u, v) = (u, uv, v)$ is a patch or not. 4

c) Show that surface of revolution is a surface. 5

6. a) Let g be a real valued differentiable function on E^3 and $M = \{(x, y, z) \in E^3 \mid g(x, y, z) = C\}$ is a surface in E^3. Prove that the gradient vector field $\nabla g = \sum_{i=1}^{3} \frac{\partial g}{\partial x_i} U_i$ is a non-vanishing normal vector field on M. 7

b) Parametrize the surface obtained by revolving the curve $C : (z - 2)^2 + y^2 = 1$ around y-axis. 3

c) Let $X : D \to M$ be a patch in a surface M and let ϕ and γ be 1-form and 2-forms respectively on M. Prove that :

i) $X^*(\phi) = \phi(X_u)du + \phi(X_v)dv$

ii) $X^*(\gamma) = \gamma(X_u, X_v)du\,dv$. 6

7. a) Define shape operator of a surface. Find the shape operator of :

i) a sphere of radius r

ii) a cylinder in E^3. 6

b) With usual notations prove :

$$K = \frac{\begin{vmatrix} SV \cdot V & SV \cdot W \\ SW \cdot V & SW \cdot W \end{vmatrix}}{\begin{vmatrix} V \cdot V & V \cdot W \\ W \cdot V & W \cdot W \end{vmatrix}}, H = \frac{\begin{vmatrix} SV \cdot V & SV \cdot W \\ V \cdot V & W \cdot W \end{vmatrix} + \begin{vmatrix} V \cdot V & V \cdot W \\ SW \cdot V & SV \cdot W \end{vmatrix}}{2\begin{vmatrix} V \cdot V & V \cdot W \\ W \cdot V & W \cdot W \end{vmatrix}}.$$ 6

c) Show that :

i) a plane in E^3.

ii) a cylinder in E^3 are flat surfaces. 4

8. a) Compute K and H hence K_1 and K_2 for the surface $M : Z = xy$, where K, H, K_1 and K_2 have their usual meaning. 6

b) Determine the geodesics in a cylinder. 4

c) Prove that the tangent vector $V = v_1 X_u + v_2 X_v$ is a principal vector if and only if $\begin{vmatrix} v_2^2 & -v_1 v_2 & v_1^2 \\ E & F & G \\ e & m & n \end{vmatrix} = 0$. 6